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MINIMUM-TIME SHIP ROUTING BY CALCULUS OF VARIATIONS METHODS

HARRY D. HAMILTON

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BY CALCULUS OF VARIATIONS METHODS

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by

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
METEOROLOGY

United States Naval Postgraduate School
Monterey, California

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ABSTRACT

The manual method of minimum-time ship routing has proved so successful over the last several years that ever increasing numbers of ships' captains are requesting this service. The desirability of the electronic computer for the computation is therefore readily apparent. In this investigation the minimum-time route is determined by calculus of variations, rather than the conventional manual technique. The present method consists of solving the associated Euler equation by numerical integration on the Control Data Corporation model 1604 digital computer. In case 1 the ship's speed is assumed to be primarily a function of position; and in case 2 the ship's speed is taken to be a function of its direction as well as position. Case 2 is recommended for operational adaptation.

The writer wishes to express his appreciation for the assistance given him by Professor George J. Haltiner of the United States Naval Postgraduate School in this study.

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TABLE OF SYMBOLS

T = Time, hours

V = Ship's speed, knots

ψ = $1/V$, hours per nautical mile

ds = Element of arc length along the track

y' = dy/dx

y'' = d^2y/dx^2

y_i = y value at the i^{th} grid point

y_{i+1} = y value at the $(i+1)$ grid point

y_{i-1} = y value at the $(i-1)$ grid point

D = Finite increment in the x-direction

$\hat{y}' = \dfrac{y_{i+1} - y_{i-1}}{2D}$

$\hat{y}'' = \dfrac{y_{i+1} - 2y_i + y_{i-1}}{D^2}$

H = Wave height, in feet

β = Direction from which waves are coming

α = Heading of ship

θ = Rotation angle between cartesian coordinates of wave model (x_w, y_w) and of ship routing (x,y)

1. Introduction

A manual method of minimum-time ship routing was developed and tested by the U.S. Navy Hydrographic Office [1]. In the investigation it was discovered that the most important parameter in retarding a ship's progress was wave action. This action manifests itself in two ways; first, the direct reduction by the resistance of the water to the ship's passage, and second, the indirect reduction through the voluntary decrease of ship's speed to reduce the violent effects of roll, pitch and heave. The two most important components of this wave action are wave height and wave direction [2].

The Fleet Weather Facilities at Alameda and at Norfolk have been very successful with the operational application of the manual method to hundreds of cases during the last several years [1].

With the advent and continued improvement of electronic computers, there has been a simultaneous increase in their application to scientific and engineering problems. One such example is the processing of meteorological data and the subsequent forecasts.

The application of the electronic computer to the minimum-time ship routing problem appears desirable as ever-increasing numbers of ships' captains are requesting this service. The first natural impulse would be to adapt the present manual method to the computer. However there is a branch of mathematics called calculus of variations which deals with the problem of minimizing or maximizing

the value of an integral by the determination of an appropriate function, in this case the proper route which will minimize the time required for a ship to travel between two ports of call.

Under quite general conditions it may be shown that the function y(x) necessary to minimize or maximize the integral

$$\int_{x_1}^{x_2} F(x, y, y') \, dx \tag{1}$$

must fulfill the Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \tag{2}$$

The boundary conditions applied here are that the y values at x_1 and x_2 are fixed [3].

The time T required for a ship or other vehicle to transverse a track is given by the line integral

$$T = \int_{S_1}^{S_2} \frac{ds}{V} \tag{3}$$

In cartesian coordinates ds may be replaced by $\left[dx^2 + dy^2\right]^{1/2}$, and (1) may now be put in the form

$$T = \int_{x_1}^{x_2} \psi \left[1 + (y')^2\right]^{1/2} dx \tag{4}$$

Where: $\psi \left[1 + (y')^2\right]^{1/2} = F(x, y, y')$.

The Euler equation (2), which must be satisfied for a minimum, becomes

$$\frac{d}{dx}\frac{\partial\left[\psi(1+y'^2)^{1/2}\right]}{\partial y'} - \frac{\partial\left[\psi(1+y'^2)^{1/2}\right]}{\partial y} = 0 \tag{5}$$

Several methods are available for the solution of the Euler equation. One possibility is to treat it as an initial-value problem. For a given initial heading of the ship there will be an error in the arrival point as determined by this method from the desired destination. Thus a first estimate is made for the initial heading and the subsequent error is determined. A second initial heading is then chosen in such a way as to reduce the corresponding error at the final point. Thus the initial heading is successively modified until the destination error is within acceptable limits of accuracy.

A relaxation method is used in this study and will be described in a later section.

Richard W. James [2] found that theoretical equations representing ships' speed under various seas did not fit observed data taken from ships' logs. One of the many reasons given for this discrepancy was the human factor of voluntarily reducing the ship's speed in order to reduce the motions of the ship. James first classified his data according to four 90-degree sectors of relative wave direction. Waves that approached a ship within 45 degrees of:

1. the bow were classified as head waves;
2. the beam were classified as beam waves;
3. the stern were classified as following waves.

Then under each classification, wave height (feet) versus ship's speed (knots) was plotted. Using the method of least squares on the assumed straight line function $V = a + bH$, the constants "a" and "b" were determined to give the best

fit for each of the three classifications. The resulting equations are:

Head Seas	V = 18.29 - 0.529 H	
Beam Seas	V = 17.90 - 0.276 H	(6)
Following Seas	V = 18.50 - 0.168 H	

Note that zero wave height does not result in the same speed in the three equations.

2. Case 1.

In case 1 it is assumed that the ship's speed is a function of position only and not a function of ship's heading. Strictly, this implies ship's speed is a function of wave height alone. However, in practice the empirical formulas (6) give no variation in speed for a considerable change of ship's heading in their respective broad sectors of head, beam, and following seas. Hence the ship's direction may be changed appreciably without altering the empirically-determined ship's speed. It follows that in many cases no great error would arise from neglecting those terms in the Euler equation involving the variation of the ship's speed as the course changes slightly in the successive approximations. Nevertheless this more general case is considered in the next section.

If V, and therefore ψ, is a function of x and y but not y', the Euler equation (5) becomes

$$\psi y'' + \left(\frac{\partial \psi}{\partial x} y' - \frac{\partial \psi}{\partial y}\right)\left(1 + y'^2\right) = 0 \qquad (7)$$

Using central finite difference approximations $\hat{y}'$ and $\hat{y}''$ for y' and y'' in (7); the equation for y_i becomes

$$y_i = \left[\frac{y_{i+1} + y_{i-1}}{2}\right] + \left[\frac{\partial \psi}{\partial x}\left(\frac{y_{i+1} - y_{i-1}}{2D}\right) - \frac{\partial \psi}{\partial y}\right]\left[1 + \left(\frac{y_{i+1} - y_{i-1}}{2D}\right)^2\right]\frac{D^2}{2\psi} \qquad (8)$$

Since the ship's speed is expressed analytically (6), the derivatives may be expressed analytically and evaluated; or the derivatives may be approximated by finite differences. The latter method would probably be used in practical operations.

Equation (8) is solved by the relaxation method, which consists of calculating the n^{th} approximation for y_i, denoted as $y_i^{(n)}$, using the previously calculated approximations $y_{i-1}^{(n)}$ and $y_{i+1}^{(n-1)}$. The initial estimate is somewhat arbitrary and may be taken as the great circle route, for example. The initial and final points remain as fixed boundary conditions.

International steamer tracks, which are modifications of routes laid out by Lieutenant Matthew F. Maury, USN (circa 1850), are quite often selected by ships' captains when the minimal-time ship routing service is not used. These routes were the best statistical routes, based on the climatology of the storm tracks and sea conditions, to prevent damage and to decrease sailing time [1].

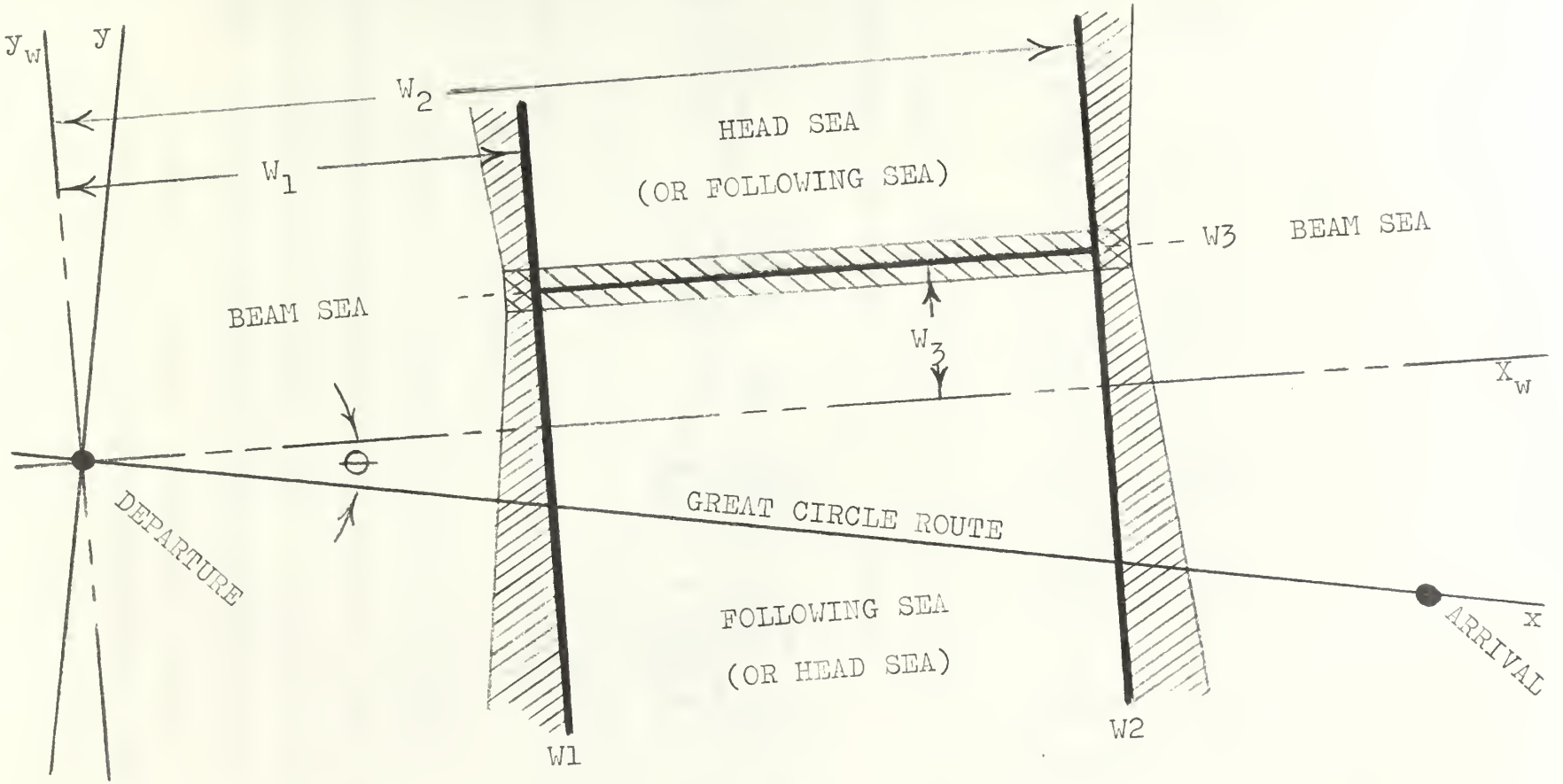
However, to keep the problem general, the initial estimate for the correct route is taken as the great circle which passes through the point of departure and the point of arrival. To take full advantage of this choice, a Lambert Conformal chart is used on which a great circle is approximately a straight line [4]. The secant projection of the Lambert Conformal chart is chosen in order to have a broader area of minimum distortion. The image scale ε, which is the dimensionless ratio of image distance to earth distance is given by

$$\varepsilon = \frac{\sin \phi_1}{\sin \phi} \left[\frac{\tan \frac{\phi}{2}}{\tan \frac{\phi_1}{2}} \right]^n$$

Where: $$n = \frac{\log \sin \phi_1 - \log \sin \phi_2}{\log \tan \frac{\phi_1}{2} - \log \tan \frac{\phi_2}{2}}$$

With the standard parallels of ϕ_1 = 30 degrees and ϕ_2 = 60 degrees, n = 0.716 [5]. Then ε = 1.0 at 30 and 60 degrees. The minimum value of ε is at 45 degrees and is approximately 0.97. By staying within a range of about 25 to 65 degrees of latitude, ε may be neglected with an error of 3% or less. The map scale does not enter into the computation of the problem.

In order to test the method, a fairly general hypothetical wave distribution was designed, as shown schematically in figure 1. In the central region between the dividing lines W1 and W2, the wave height is a function of y_w together with the maximum wave height which is constant along the dividing line W3. Outside this central area the wave height is a function of x_w and y_w, with wave height decreasing away from the central region. To avoid discontinuities narrow linear transition zones of speed were inserted. The computational procedure is programmed in such a way that the various parameters defining the model may be changed at will. Thus a large variety of sea conditions may be simulated.



CASE 1 WAVE MODEL

Figure 1

3. Case 2.

In this section the more general case where the ship's speed varies continuously with relative wave direction will be considered. Thus $\psi = \psi(x,y,y')$, and the Euler equation (5) reduces to

$$\frac{\partial \psi}{\partial x} y' - \frac{\partial \psi}{\partial y} + \left(\frac{\partial^2 \psi}{\partial y' \partial x} + \frac{\partial^2 \psi}{\partial y' \partial y} y' \right) \left[1 + y'^2 \right] + \left\{ 2 \frac{\partial \psi}{\partial y'} y' + \frac{\partial^2 \psi}{\partial y'^2} \left[1 + y'^2 \right] + \psi \left[1 + y'^2 \right]^{-1} \right\} y'' = 0 \tag{9}$$

When this equation (9) is expressed in finite difference form and solved for y_i it becomes

$$y_i = \left[\frac{y_{i+1} + y_{i-1}}{2} \right] + \left\{ \frac{\frac{\partial \psi}{\partial x} \hat{y}' - \frac{\partial \psi}{\partial y} + \left[\frac{\partial^2 \psi}{\partial y' \partial x} + \frac{\partial^2 \psi}{\partial y' \partial y} \hat{y}' \right] \left[1 + \hat{y}'^2 \right]}{\psi \left[1 + \hat{y}'^2 \right]^{-1} + 2 \frac{\partial \psi}{\partial y'} \hat{y}' + \frac{\partial^2 \psi}{\partial y'^2} \left[1 + \hat{y}'^2 \right]} \right\} \frac{D^2}{2} \tag{10}$$

In order to allow for a continuous change of ship's speed with relative wave direction, James' three equations (6) may be approximated by

$$V = 18.23 - 0.302 + 0.189 \cos (\beta - \alpha) H \tag{11}$$

If a better approximation with head and beam wave conditions is desired, with a slight loss of accuracy with following wave conditions, then (11) may be modified to

$$V = 18.23 - 0.302 + 0.225 \cos (\beta - \alpha) H \tag{12a}$$

On the other hand, a better approximation with following and beam wave conditions, with a slight loss of accuracy with head wave conditions, is

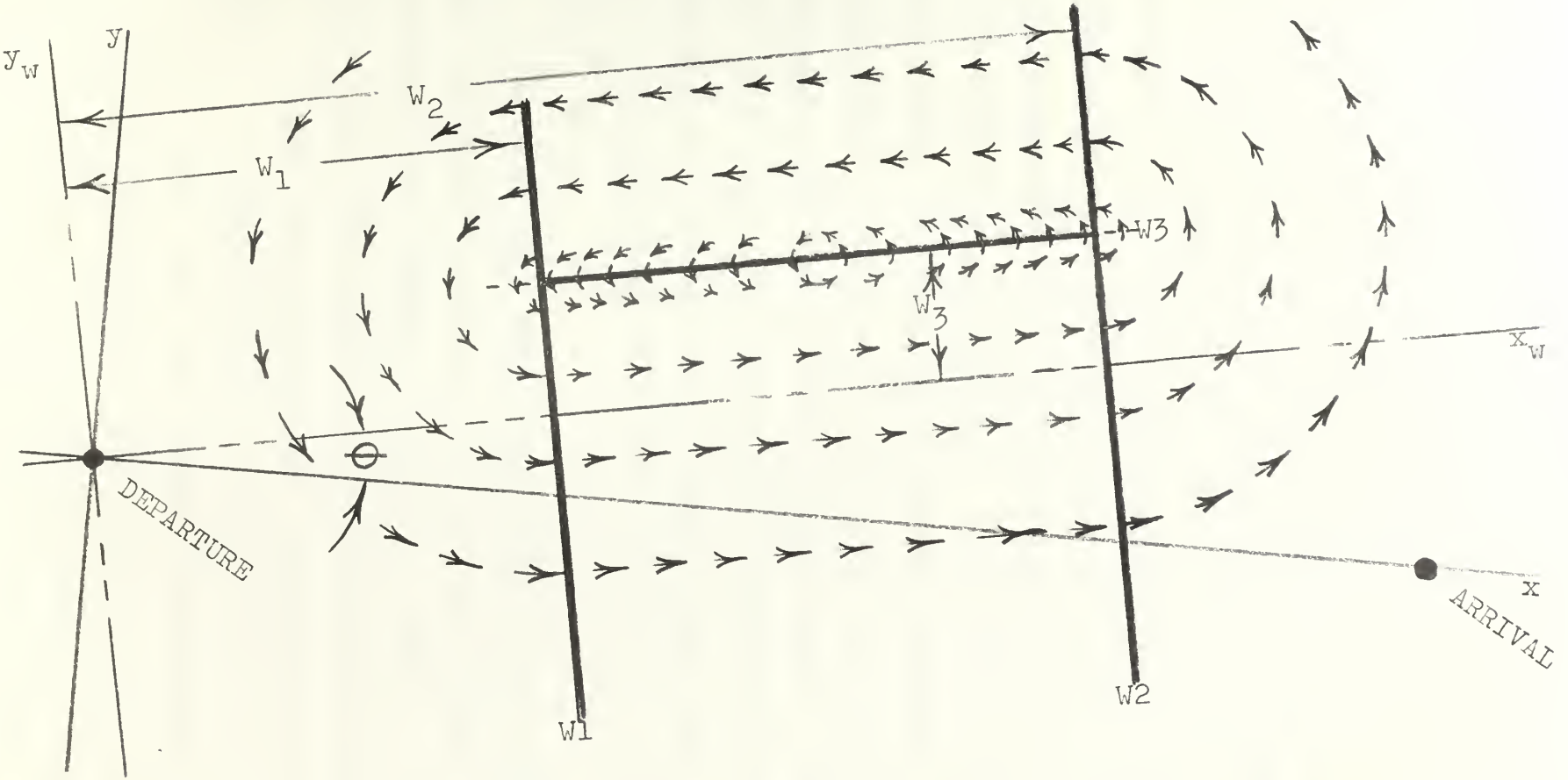
$$V = 18.23 - 0.302 + 0.153 \cos(\beta - \alpha) H \qquad (12b)$$

The last equation was used for the results in table 1; however the three constants in order were rounded to 18.0, 0.30 and 0.15, respectively. With these values, the three cases of head, beam and following seas are represented by

$$\begin{aligned} &\text{Head Seas} && V = 18.0 - 0.45 H \\ &\text{Beam Seas} && V = 18.0 - 0.30 H \\ &\text{Following Seas} && V = 18.0 - 0.15 H \end{aligned} \qquad (13)$$

In order to test the effect of including the variation of ship's speed with its direction in the Euler equation, only the addition of wave direction at every point to the previously-designed wave model was necessary, as shown schematically in figure 2. Outside the central region of the model the wave direction follows a circular change until it returns to the central area. A transition zone is needed only to separate head and following seas. This is accomplished by a rapid change of wave direction between these two areas. The remaining short line of discontinuity of speed is positioned at the start of each specific problem near the center of the transition zone, but in such a position that it will not become involved in the subsequent calculations.

Other conditions remain as before.



CASE 2 WAVE MODEL

Figure 2

Results.

Case 1 and case 2 results were both obtained with initial conditions as specified below (see figures 1 and 2). The great-circle distance between departure and arrival points was 4,800 n mi. The finite increment in the x-direction D was 60 n mi. The rotation angle θ, was zero; thus the two coordinate axes coincided. W_1 was 1200 and W_2 was 3600 n mi. The general wave-directional pattern was cyclonic, that is, head seas above the great circle (positive y-direction) and following seas below the great circle (negative y-direction). All initial y values were taken as zero along the great circle; therefore the maximum displacements shown in the table are the perpendicular distances away from the great-circle route. The maximum wave height was varied from zero to thirty feet at one-foot increments. Wave height decreased at the rate of 0.03 feet per n mi with respect to x and 0.06 feet per n mi with respect to y.

The calculated results of cases 1 and 2 varied smoothly, therefore only the results obtained at five-foot increments of maximum wave height are listed in table 1. The results in the table are rounded to the nearest one-tenth unit.

In case 1 the great-circle estimate went through the middle of the head-following linear transition zone, that is, W_3 was zero n mi. The apparent discrepancy for zero wave-height is a result of the unequal speeds from the James' equation (6) for zero wave height, and the passage through a linear transition zone of the model.

In case 2 the great-circle estimate went through on the upper limit of the transition zone, that is, head seas were encountered throughout the central region, with W_3 set at minus one n mi.

In comparing the results of cases 1 and 2, the minimum times and maximum displacements should be compared rather than the great-circle times, since the greatest differences between equations (6) and (12b) or (13) occur with head seas.

In cases 1 and 2 the convergence to the minimum time varied in a nonlinear manner with the rate of convergence decreasing as the minimum-time track was approached.

To make certain that the calculated results were not just relative minimums due to some unforeseen combination of speed equations and wave model, initial estimates other than the great circle were used. These were taken on both sides of the computed minimum-time track; nevertheless the final track was the same.

RESULTS OF CASE 1 AND CASE 2

CASE 1			WAVE HEIGHT (FEET)	CASE 2		
TIME (HOURS)		MAXIMUM DISPLACEMENT (N. Miles))		MAXIMUM DISPLACEMENT (N. Miles)	TIME (HOURS)	
GREAT CIRCLE	MINIMUM ROUTE				MINIMUM ROUTE	GREAT CIRCLE
264.8	263.6	-0.7	0	0.0	266.7	266.7
287.8	264.2	-114.6	5	-115.3	267.1	286.4
321.7	265.0	-196.5	10	-196.2	268.0	314.7
373.5	266.5	-280.7	15	-278.8	269.5	355.4
460.3	268.6	-361.1	20	-361.8	271.5	416.9
632.8	271.2	-443.2	25	-443.2	274.0	517.8
1156.7	274.5	-544.7	30	-527.5	277.2	713.0

Table 1

5. Conclusions.

The calculus of variations method gave good minimum-time tracks for all the runs computed. Nothing in this study was discovered that would bar operational adaptation of this method.

In many engineering and scientific problems solved by relaxation, it is agreed that the best approximation to the correct answer is the mean between the values obtained by relaxing from initial estimates on both sides of the calculated answers. However in this investigation the first calculated answer from the great circle route is well within present-day navigational standards, and no further calculation appears necessary.

In this study, case 2 took about four times as long to calculate as case 1, but since the solution time involved was still only about ten minutes on the Control Data Corporation model 1604 digital computer, the added generality seemed worth the additional time. Also since ship's heading and wave direction must be calculated in the operational use of case 1 in order to determine the proper speed, this information may as well be used to its fullest extent by utilizing case 2.

The time variation of the sea may be included in case 2 in much the same manner as the directional variation of the sea was to case 1. Fortunately the minimum-time route is generally away from active storm centers and along the fringe areas where the actual time variation of the seas is greatly reduced. Since sea conditions decay slowly, but

may build fairly rapidly, the error could be of some significance near rapidly-developing storm centers.

Thus the only real limitation to the application of this method appears to be in the quality of the forecast of sea conditions.

James' speed equations, and their approximations, were used in this study only to provide realistic values, and in an operational application each ship could have its own characteristic speed equation inserted into the program. The Lambert Conformal chart, while convenient for this experiment, is by no means necessary, and an operational program should be chosen to use the data available from forecast wave height and direction most economically.

As mentioned earlier, a further refinement of this method of minimum-time ship routing would be the inclusion of the terms in the Euler equation that involve the change of wave height and direction with time. Since the results were changed very little by the addition of the relative wave-direction terms in the Euler equation, in most situations there probably would be little change with the addition of the time-variation terms, even though they are quite numerous. The added time required for solution, as well as the extra storage of sea-condition data in or outside of the computer and the call and recall of the data for at least three time periods associated with each calculation of y_i, may prove to be prohibitive from an operational standpoint.

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